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ON A LINEAR OBJECT OBSERVATION PROBLEM*

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Minimax estimates in linear systems /1,2/ are investigated under random perturbations in the meter channel. It is proved that under reasonable assumptions they are asymptotically exact with probability one and, furthermore, that simpler estimates possessing the same properties can be constructed, effectively computable by mathematical programming methods.

1. Assume that the signal

$$y_t = \Phi_t z + w_t, \quad t \in T, \quad z \in R^n, \quad T = \{1, 2, \ldots\} \tag{1.1}$$

is being observed. Here and later lower-case Roman letters denote column-vectors and lower-case Greek letters denote row-vectors of appropriate dimensions; z is the unknown parameter vector; Φ_t are deterministic $m \times n$-matrices; w_t is a narrow-sense-stationary random process specified on the space of elementary events $\{\Omega, \Sigma, P\}$ in the phase space $\{R^m, \Lambda\}$, where Λ is the Borel σ-algebra of sets from R^m. The restriction $w_t \in W$, W is a compactum in R^m, is assumed fulfilled for all $t \in T$. The estimate $z_1(\psi, N)$ of the scalar quantity ψz, $\psi \in R^n$, $\|\psi\| = 1$, from N observations (1.1) is defined as follows:

$$z_1(\psi, N) = \inf_{\omega_t}\left\{\sum_{t=1}^{N} [\Gamma(-\omega_t \mid W) + \omega_t y_t]: \sum_{t=1}^{N} \omega_t \Phi_t = \psi\right\}, \quad \Gamma(\omega \mid W) = \sup\{\omega w: w \in W\} \tag{1.2}$$

is the support function of set W, $z_1(\psi, N)$ is the value of the support function of the domain compatible with the signal $y_1, \ldots, y_N$ being realized in the case of indeterminate noise and of a convex set W /2/. The derivation of (1.2) is based on the standard duality relations of convex analysis. We state several conditions.

Condition 1. There exist $M > 0$, a positive integer $l \geqslant 1$ and a partitioning $\{J_k\}$ of set T into collections J_k, $k = 1, 2, \ldots$, of l indices each, such that: 1) $J_i \cap J_j = \phi$, $i \neq j$; 2) if $i, j \in J_k$, $i \neq j$, then $|i - j| \geqslant k$; 3) for any $k \geqslant 1$ the system of equations

$$\sum_{i \in J_k} \omega_i \Phi_i = \psi \tag{1.3}$$

has a solution ω_i such that

$$\|\omega_i\| \leqslant M, \quad i \in J_k \tag{1.4}$$

By S^t we denote the semigroup of measure-preserving transformations of a shift, connected with process $\{w_t, t \in T\}$. If U^t is the semigroup of transformations of the shift of random variables, corresponding to S^t, then /3/ $w_t = U^t w_0$.

Condition 2. For any ϑ, $\|\vartheta\| = 1$ and $\varepsilon > 0$

$$P\{\Gamma(-\vartheta \mid W) + \vartheta w_0 < \varepsilon\} \geqslant p(\varepsilon) > 0$$

Condition 3. The process $\{w_t, t \in T\}$ is regular /3/.

Condition 4. There exists $\varepsilon_* > 0$ such that for all ϑ, $\|\vartheta\| = 1$, the distribution function of the random variable $\eta = \Gamma(-\vartheta \mid W) + \vartheta w_0$ is continuous on $(0, \varepsilon_*)$.

Condition 5. The random process $\{w_t, t \in T\}$ is a process with mixing /3,4/.

Condition 5'. The random process $\{w_t, t \in T\}$ satisfies the weak mixing condition /4/.

Theorem 1. If Conditions 1, 2, 3 or 1, 2, 4, 5 (5') are fulfilled, then estimate (1.2) is consistent and $z_1(\psi, N) \to \psi z$ as $N \to \infty$ with probability one.

Estimate (1.2) is nonlinear. We now assume that Condition 1 has been fulfilled and we choose ω_t in the following manner: the ω_t satisfy (1.3) and (1.4) for all $t \in J_k$, $k = 1, 2, \ldots$,

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and are arbitrary for the remaining $t \in T$. We define a linear estimate $z_2(\psi, N)$ of the quantity ψz:

$$z_2(\psi, N) = \inf_{\lambda_t}\left\{\sum_{t=1}^{N} \lambda_t [\Gamma(-\omega_t | W) + \omega_t y_t] : \sum_{t=1}^{N} \lambda_t \omega_t \Phi_t = \psi, \quad \lambda_t \geqslant 0\right\} \tag{1.5}$$

The next theorem is our main result:

Theorem 2. If conditions 2, 3 or 2, 4, 5 (5') are fulfilled, then estimate $z_2(\psi, N)$ is consistent and $z_2(\psi, N) \to \varphi z$ as $N \to \infty$ with probability one.

Proof of Theorem 1. Substituting the expression for y_t, $t \in T$, into (1.2) and (1.5), we can obtain

$$z_i(\psi, N) = \psi z + \gamma_i(\psi, N), \; i \subset 1, 2$$

$$\gamma_1(\psi, N) = \inf_{\omega_t}\left\{\sum_{t=1}^{N} [\Gamma(-\omega_t | W) + \omega_t w_t] : \sum_{t=1}^{N} \omega_t \Phi_t = \psi\right\}$$

$$\gamma_2(\psi, N) = \inf_{\lambda_t}\left\{\sum_{t=1}^{N} \lambda_t [\Gamma(-\omega_t | W) + \omega_t w_t] : \sum_{t=1}^{N} \lambda_t \omega_t \Phi_t = \psi, \quad \lambda_t \geqslant 0\right\}$$

Since $0 \leqslant \gamma_1(\psi, N) \leqslant \gamma_2(\psi, N)$, the assertion of Theorem 1 follows from Theorem 2.

Proof of Theorem 2. Let $J_i \in \{1, 2, \ldots, N\}$, $i = 1, 2, \ldots, K$, $L \geqslant 1$. Then

$$\gamma_2(\psi, N) \leqslant \inf_{\lambda_t}\left\{\sum_{t=1}^{K} \lambda_t \sum_{i \in J_t} [\Gamma(-\omega_i | W) + \omega_i w_i] : \lambda_t \geqslant 0, \; \sum_{t=1}^{K} \lambda_t = 1\right\} \leqslant \min\left\{\sum_{i \in J_t} [\Gamma(-\omega_i | W) + \omega_i w_i] : L \leqslant t \leqslant K\right\}$$

By $A_i(\varepsilon, \omega)$ and $B_k(\varepsilon)$ we denote the following events:

$$A_i(\varepsilon, \omega) = \{\Gamma(-\omega \mid W) + \omega w_i < \varepsilon\}$$

$$B_k(\varepsilon) = \left\{\sum_{i \in J_k} [\Gamma(-\omega_i | W) + \omega_i w_i] \geqslant \varepsilon\right\}$$

We see that

$$P(B_k(\varepsilon)) \leqslant 1 - P(\bigcap_{i \in J_k} A_i(\varepsilon l^{-1}, \omega_i))$$

We now assume the fulfillment of Conditions 2 and 3. Then $L \geqslant 1$ exists and for all $k \geqslant L$

$$P(\bigcap_{i \in J_k} A_i(\varepsilon l^{-1}, \omega_i)) \geqslant {}^1/_2 \prod_{i \in J_k} P(A_0(\varepsilon l^{-1}, \omega_i)) \geqslant {}^1/_2 (p(\varepsilon(lM)^{-1}))^l > 0$$

Comparing the last two inequalities, we obtain

$$P(B_k(\varepsilon)) \leqslant 1 - {}^1/_2 \; (p(\varepsilon(lM)^{-1}))^l < 1$$

From the resularity of w_t it follows that for any $r \geqslant 1$ there exists K_0 (respectively, N_0) such that for all $K \geqslant K_0$ $(N \geqslant N_0)$

$$P(\bigcap_{k=L}^{K} B_k(\varepsilon)) \leqslant 2(1 - {}^1/_2(p(\varepsilon(lM)^{-1}))^l)^r$$

Since

$$P\left\{\min\left\{\sum_{i \in J_t} [\Gamma(-\omega_i | W) + \omega_i w_i] : L \leqslant t \leqslant K\right\} < \varepsilon\right\} \geqslant 1 - P(\bigcap_{k=L}^{K} B_k(\varepsilon))$$

we finally obtain that for $N \geqslant N_0$

$$P\{|z_2(\psi, N) - \psi z| < \varepsilon\} \geqslant 1 - 2(1 - {}^1/_2(p(\varepsilon(lM)^{-1}))^l)^r$$

i.e., $z_2(\psi, N)$ converges in probability to ψz. Convergence with probability one follows simply from the monotonicity of $z_2(\psi, N)$ with resepct to N.

We now note that the regularity of process w_t was used only when justifying for all sufficiently small $\varepsilon > 0$ the equality

$$\lim_{i\to\infty} \sup_{\|\omega\|\leqslant M} | P (S^i A_0 (\varepsilon, \omega) \cap C) - P (A_0 (\varepsilon, \omega)) P (C) | = 0 \tag{1.6}$$

where $C \in \Sigma (T)$ is the σ-algebra induced by events of the form $\{w_t \in G\}$, $G \in \Delta$, $t \in T$. The latter is valid, in particular, if Conditions 4 and 5 are fulfilled. We prove this fact by contradiction.

Let $\varepsilon_0 > 0$ and ω_i^*, $i = 1, 2, \ldots$, exist such that

$$| P (S^i A_0 (\varepsilon, \omega_i^*) \cap C) - P (A_0 (\varepsilon, \omega_i^*)) P (C) | \geqslant \varepsilon_0 \tag{1.7}$$

Since $\| \omega_i^* \| \leqslant M$, without loss of generality we can take it that $\lim_{i\to\infty} \omega_i^* = \omega_*$ and $\| \omega_* \| \leqslant M$. Therefore, for sufficiently large i we have $A_0 (\varepsilon, \omega_i^*) \supset A_0 (\varepsilon, \omega_*)$. Consequently,

$$\begin{aligned} | P (S^i A_0 (\varepsilon, \omega_i^*) \cap C) - P (A_0 (\varepsilon, \omega_i^*)) P (C) | \leqslant \\ | P (S^i A_0 (\varepsilon, \omega_*) \cap C) - P (A_0 (\varepsilon, \omega_*)) P (C) + \\ 2 | P (A_0 (\varepsilon, \omega_i^*) - P (A_0 (\varepsilon, \omega_*)) | \end{aligned} \tag{1.8}$$

The latter inequality contradicts (1.7) since $\lim_{i\to\infty} | P (S^i A_0 (\varepsilon, \omega_*) \cap C) - P (A_0 (\varepsilon, \omega_*)) P (C) | = 0$ for processes with mixing, while from the continuity of $\Gamma(\cdot | W)$, Helly's theorem and Condition 4 it follows that for sufficiently small $\varepsilon > 0$

$$\lim_{i\to\infty} P (A_0(\varepsilon, \omega_i^*)) = P (A_0 (\varepsilon, \omega_*))$$

The validity of equality (1.6) has been established.

Condition 5 can be replaced by Condition 5'. To prove this it is enough to make use of the simple properties of weak mixing /4/. If observations (1.1) are independent, we can waive property 2) in Condition 1.

As a discussion of the results we note the following.

1°. To form the estimates $z_i (\psi, N)$, $i = 1, 2$, we do not need to know the distributions of the random variables w_t on set W or any characteristic of such a distribution. The consistency of the estimates in the case of independent observations obtains under the fulfillment only of Condition 2, which can be interpreted as the start of a trial run of the meter.

2°. The estimates $z_i (\psi, N)$, $i = 1, 2$, are upper bounds for ψz and do not grow monotonically as $N \to \infty$. The estimates $-z_i (-\psi, N)$, $i = 1, 2$, obviously, are monotonically-nondecreasing lower bounds for ψz. Consequently, Theorems 1 and 2 are valid for estimates of the form $\frac{1}{2} (z_i (\psi, N) - z_i (-\psi, N)), i = 1, 2$, whose application yields a convenient stopping rule with respect to the accuracy achieved.

3°. When constructing the estimate $z_2 (\psi, N)$ the choice of ω_t, $t \in T$, can be made before the start of the mearurements, and it depends solely on the properties of the matrix Φ_t. If necessary, several ω_t can be chosen for each $t \in T$, having set, for example, $\bar{\omega}_t = \{\pm e_j, j = 1, 2, \ldots, m\}$ (the e_j-th unit vector). Obviously, all the ω_i, $i \in J_t$, satisfying (1.3) can be represented as a linear combination of vectors from $\bar{\omega}_t$ with positive coefficients. Formally this leads to an estimate of form (1.5) with due regard to the ambiguity in the choice of ω_t.

4°. Estimate $z_2 (\psi, N)$ is formed from the solution of a linear programming problem in standard form. Furthermore, as N grows it is easy to make a recurrent transition from problem to problem since the addition of one measurement signifies the writing in of a new column in the matrix of constraints of problem (1.5). In this sense the construction of estimate $z_2 (\psi, N)$ is essentially simpler than that of $z_1 (\psi, N)$, which, as already noted, is typical for the case of indeterminate noise of form (1.2).

Example 1. Suppose that we are measuring the scalar signal $y_t = z + w_t$, $w_t \in [-1, 1]$ is a random process with independent values, and for all $\varepsilon > 0$ let $P\{1 \pm w_t < \varepsilon\} \geqslant p(\varepsilon) > 0$, $\psi = +1$. In this case

$$z_1 (\psi, N) = 1 + \min \{y_t : 1 \leqslant t \leqslant N\}$$

If w_t is distributed uniformly on $[-1, 1]$, then $z_1 (\psi, N)$ coincides with one of the maximum likelihood estimates, while the estimate $\frac{1}{2} (z_1 (\psi, N) - z_1 (-\psi, N)) = \frac{1}{2} (\max_t y_t + \min_t y_t)$ is unbiased and effective.

2. Let us consider the observation problem /1/ for a linear dynamic system

$$x^{\cdot} = Ax, \; t \geqslant 0 \tag{2.1}$$

The vectors

$$y_k = Bx (t_k) + w_k \tag{2.2}$$

are measured at discrete instants t_k, $k = 1, 2, \ldots$. Here A and B are constant matrices of dimensions $n \times n$ and $m \times n$, respectively. It we pose the problem of determining the initial state

$x(0) = z \in R^n$, then observations (2.2) reduce to process (1.1) with $\Phi_k = B \exp At_k$. Let us assume that the observations are made after equal time intervals $\tau > 0, t_k = k\tau, k = 1, 2, \ldots$, and rank $\| B^*, A^*B^*, \ldots, A^{*(n-1)} \cdot B^* \| = n$ (here the asterisk denotes transposition). This singifies that the system is completely observable with respect to output (2.2) when $w_k \equiv 0, k = 1, 2, \ldots$.

A principal feature determining the possibility of applying Theorems 1 and 2 for this case is the verification of the fulfillment of Condition 1. Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of matrix A, of multiplicities $k_1, \ldots, k_r$ respectively. Further, let $h_{ij}, j = 1, 2, \ldots, k_i$, be the vectors of a series relative to matrix A with eigenvalues λ_i, i.e.,

$$Ah_{i1} = \lambda_i h_{i1}, \quad Ah_{i2} = \lambda_i h_{i2} + h_{i1}, \ldots, \quad Ah_{ik_i} = \lambda_i h_{ik_i} + h_{ik_i - 1}$$

The vectors $h_{ij}, j = 1, 2, \ldots, k_i, i = 1, 2, \ldots, r$, form a basis in R^n, in which matrix A has a Jordan form.

We shall take it that

$$\varkappa_1 = \operatorname{Re} \lambda_1 = \ldots = \operatorname{Re} \lambda_{r_1} < \varkappa_2 = \operatorname{Re} \lambda_{r_1+1} = \ldots = \operatorname{Re} \lambda_{r_1+r_2} < \varkappa_3 = \ldots < \varkappa_g = \ldots = \operatorname{Re} \lambda_r$$

and, in addition, $\varkappa_\nu < 0 \leqslant \varkappa_{\nu+1}$. The following basic statement is valid.

Lemma 1. There exists $\tau > 0$ such that when $\Phi_k = B \exp Ak\tau$ condition 1 with $l = n$ is fulfilled for all $\psi, \| \psi \| = 1$, satisfying the condition

$$\psi h_{ij} = 0, \quad j = 1, 2, \ldots, k_i, \quad i = 1, 2, \ldots, \sum_{s=1}^{\nu} r_s \tag{2.3}$$

The proof of this lemma is rather cumbersome and we omit it here. We merely make certain remarks. The prohibited values of the discretization parameter τ are determined by the periods typical of system (2.1). If matrix A has multiple pure imaginary eigenvalues, then the uniform boundedness of the solutions of (1.3) can be achieved by selecting $J_k = \{k_i, i = 1, 2, \ldots, n\}$ such that the quantity $k_i | k_i - k_j |^{-1}$ is bounded for all $i, j = 1, 2, \ldots, n, i \neq j, k = 1, 2, \ldots$. The set of vectors ψ satisfying (2.3) in the sense defined is sufficient for solving the problem posed, since

$$\psi \exp (At) z = \psi_1 \exp (At) z + \delta(t)$$
$$| \delta(t) | \leqslant C \exp \varkappa_\nu t \to 0, \quad t \to +\infty, \quad C = \text{const}$$

where ψ_1 satisfies (2.3) and $\psi_1 \neq 0$ if only $\nu < g$.

Example 2. Let

$$A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \qquad B = \| 1, 0 \|, \qquad \tau = 1$$

$\{w_t, t \in T\}$ is the same as in Example 1. For $J_k = \{k_1, k_2\}$ and $\psi = \| \psi_1, \psi_2 \|$ the solution of (1.3) is determined as follows:

$$\omega_{k_1} = \frac{\psi_1 k_2 - \psi_2}{k_2 - k_1}, \qquad \omega_{k_2} = \frac{\psi_2 - \psi_1 k_1}{k_2 - k_1}$$

From (1.5) we obtain estimates for the initial position $x_1(K)$ and velocity $x_2(K)$:

$$x_1(K) = \min_{1 \leqslant i \leqslant K} \frac{k_2 (1 + y_{k_1}) - k_1 (1 - y_{k_2})}{k_2 - k_1}, \qquad \psi = \| 1, 0 \|$$
$$x_2(K) = \min_{1 \leqslant i \leqslant K} \frac{(1 + y_{k_2}) + (1 - y_{k_1})}{k_2 - k_1}, \qquad \psi = \| 0, 1 \|$$

The consistency of $x_1(K)$ obtains if the ratio $(k_1 + k_2) / (k_2 - k_1)$ is bounded. Obviously, this can be ensured.

Example 3. Let

$$A = \begin{Vmatrix} 0 & 1 \\ -\omega^2 & -2\delta \end{Vmatrix}, \qquad B = \| 1, 0 \|$$

and w_t be the same as in the preceding example. If $\gamma^2 = \omega^2 - \delta^2 > 0$, then matrix A has the simple eigenvalues $\lambda_{12} = -\delta \pm i\gamma$. If $\delta > 0$, then $\operatorname{Re} \lambda_{12} < 0$ and $\| x(t) \| \leqslant C \exp(-\delta t)$, where $x(t)$ satisfies system (2.1). Let $\delta \leqslant 0$. Then for $\psi = \| \psi_1, \psi_2 \| \; \omega_{k_1} = (\Psi_1 S_2 - \Psi_2 S_1) \mu^{-1}$

$$\omega_{k_2} = -[\Psi_1 (C_2 + \delta S_2) + \Psi_2 (C_1 + \delta S_1)] \mu^{-1}$$
$$\Psi_j = \psi_j \exp \delta k_j \tau, \; S_j = \sin \gamma k_j \tau, \; C_j = \gamma \cos \gamma k_j \tau$$
$$j = 1, 2, \; \mu = \gamma \sin \gamma (k_2 - k_1) \tau$$
$$J_k = \{k_1, k_2\}, \; k_2 = k_1 + 1, \; k = 1, 2, \ldots$$

Estimate (1.5) has the form

$$z_2(\psi, K) = \min_{1 \leqslant k \leqslant K} \{(|\omega_{k_1}| + \omega_{k_1} y_{k_1}) + (|\omega_{k_2}| + \omega_{k_2} y_{k_2})\}$$

The latter is consistent if only $\sin \gamma\tau > 0$.

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